



# Technology and Decision Rules in the Theory of Investment Behavior

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# TECHNOLOGY AND DECISION RULES IN THE THEORY OF INVESTMENT BEHAVIOR

DALE W. JORGENSON

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## 1. INTRODUCTION

The theory of production — including the description of technology, the conditions for producer equilibrium, and the response of variables determined in the theory to changes in parameters — may be developed in a form that abstracts from specific interpretations, as in Hicks, Samuelson, and Debreu.<sup>1</sup> In the intertemporal theory of production, each commodity is distinguished by point of time. As an illustration, an hour of labor today and an hour of labor tomorrow are treated as distinct commodities. Given this interpretation, the intertemporal theory of producer behavior is formally analogous to the standard atemporal theory. This analogy has been developed extensively by Fisher, Lindahl, Hicks, and Malinvaud.<sup>2</sup>

The special character of the intertemporal theory of production arises from more detailed specifications of technology, permitting more precise characterization of producer behavior. The most important specialization of technology results from introduction of capital as a factor of production. Although the central concepts of capital theory — capital assets, capital services, and investment — are inessential to the development of the intertemporal theory of production, they provide additional structure that permits a much more detailed analysis of producer behavior.

First, in the theory of capital the service of a durable good at any point of time is a fixed proportion of the output of the same service at any other point of time. These proportions describe the relative efficiencies of a durable good of different ages. Second, the

1. J. R. Hicks, *Value and Capital*, 2nd ed. (Oxford: Oxford University Press, 1946); P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge: Harvard University Press, 1947); and G. Debreu, *Theory of Value* (New York: John Wiley and Sons, 1959).

2. I. Fisher, *The Theory of Interest* (New York: A. M. Kelley, 1954); E. Lindahl, "The Place of Capital in the Theory of Prices," in E. Lindahl, *Studies in the Theory of Money and Capital* (London: Allen and Unwin, Ltd., 1939); Hicks, *op. cit.*; E. Malinvaud, "Capital Accumulation and Efficient Allocation of Resources," *Econometrica*, XXI (April 1953), 233-68.

services of durable goods acquired at different points of time are perfect substitutes in production. Accordingly, the flow of capital services to the activity of production depends on the stock of capital. The assumptions of perfect complementarity and perfect substitution underlying the model of durable capital are highly restrictive, but the practical importance of this model is very great, as Lindahl and Malinvaud have pointed out.<sup>3</sup>

The special character of technology associated with capital as a factor of production results from the fact that production possibilities at each point of time depend only on the accumulated stock of investment goods at that point of time and are otherwise independent of past productive activity. We refer to this class of technologies as recursive; in recursive technologies production possibilities at any point of time may depend on accumulated stocks of the produced means of production, but are otherwise independent of past activity.

For recursive technologies the intertemporal theory of producer behavior may be radically simplified. Production and investment decisions at each point of time depend only on production possibilities at that point of time. Since production and investment decisions may depend on accumulated stocks of investment goods, we refer to the associated decision rules as recursive. For a recursive technology the objective of the firm may be interpreted as maximization of goodwill subject to initial capital stocks at each point of time. Recursive technologies and the associated decision rules provide the basis for optimal control theory.

In the classical theory of capital, as developed by Wicksell, Lindahl, and Malinvaud,<sup>4</sup> the installation of investment goods is assumed to be free of restriction. Any quantity of investment goods can be installed (or removed) at a given point of time. Under this assumption the possibilities for productive activity are independent of the accumulated stock of capital. This specialization of technology results in a further radical simplification of the intertemporal theory of producer behavior. The decisions of the producing unit at each point of time are independent of accumulated stocks; the intertemporal decision process can be decomposed into separate decisions taking place at distinct points of time. Accordingly, we refer to these technologies and the associated decision rules as decom-

3. Lindahl, *op. cit.*; Malinvaud, *op. cit.*

4. K. Wicksell, "The Theory of Production and Distribution," in L. Robbins, ed., *Lectures on Political Economy*, Vol. I, Part II, trans. E. Classen (London: Routledge and Kegan Paul, 1934), esp. pp. 144-206; Lindahl, *op. cit.*; Malinvaud, *op. cit.*

posable. Under a decomposable technology the objective of the firm is simply to maximize profit at each point of time.

The notion of a decomposable technology was formalized by Malinvaud.<sup>5</sup> Decomposable technologies include the durable goods model of Walras, Åkerman, and Wicksell<sup>6</sup> and the inventory model underlying the Austrian theory of capital of Böhm-Bawerk and Wicksell.<sup>7</sup> The notion of a decomposable decision rule was introduced by Malinvaud and interpreted as the consequence of maximization of profit at each point of time.<sup>8</sup>

## 2. TECHNOLOGY

We consider two alternative specifications of technology, both involving capital services as a factor of production. We retain the assumption that capital services from a durable good are perfectly complementary at different points of time. Further, we assume that the services of durable goods acquired at different points of time are perfect substitutes in production at any point of time. However, we do not retain the classical assumption that installation of durable goods is free of restriction. We first develop the theory of optimal capital accumulation with restriction on the installation of investment goods and then specialize the theory to the classical characterization of technology by eliminating this restriction.

Two types of restrictions on the installation of investment goods have been treated in the theory of investment behavior. First, the rate of installation of investment goods may be irreversible so that investment must be nonnegative. Optimal investment policy under this type of restriction has been analyzed in detail by Arrow and others.<sup>9</sup> Second, investment may be associated with the use of internal resources of the productive unit. Optimal policy under this type of restriction has been analyzed by Lucas, Uzawa, and others<sup>1</sup>

5. Malinvaud, *op. cit.*

6. L. Walras, *Elements of Pure Economics*, trans. W. Jaffé (Homewood, Ill.: Richard D. Irwin, 1954); G. Åkerman, *Realkapital and Kapitalzins* (Stockholm: Centraltryckeriet, 1923); and K. Wicksell, "A Mathematical Analysis of Dr. Åkerman's Problem," in *Lectures*, Vol. I, pp. 274-99.

7. E. R. Böhm von Bawerk, *The Positive Theory of Capital*, trans. W. Smart (London: Macmillan, 1891); and Wicksell, *Lectures*, Vol. I, Part II, "The Theory of Production and Distribution," esp. pp. 144-206.

8. Malinvaud, *op. cit.*, p. 253.

9. See K. J. Arrow, "Optimal Capital Policy with Irreversible Investments," in J. N. Wolfe, ed., *Value, Capital, and Growth* (Chicago: Aldine Publishing Company, 1968), pp. 1-20, and the references given there.

1. R. Lucas, "Adjustment Costs and the Theory of Supply," *Journal of Political Economy*, LXXV (Aug. 1967), 321-34; H. Uzawa, "Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth," *Journal of Political Economy*, LXXVII (July/Aug. 1969), 628-52.

as a formalization of the investment theory of Lerner,<sup>2</sup> and Eisner and Strotz.<sup>3</sup> We derive and characterize optimal investment policy under restrictions on the installation of investment goods including both irreversibility and the use of internal resources. Pontryagin's maximum principle provides a natural framework for unifying these disparate strands in the theory of investment.<sup>4</sup>

To simplify the presentation, we consider a description of technology involving only one product at any point of time, denoted  $Q(t)$ , and only one factor of production other than capital, say labor services, denoted  $L(t)$ . Finally, we assume that capital services are a factor of production; the quantity of capital services is denoted  $K(t)$ . The flow of labor services is measured in man-hours and the flow of capital services is measured in machine-hours. The intertemporal theory of producer behavior is easily extended to any number of products and factors of production.

The level of capital services provided to the productive process may be altered by investment activity. For *recursive* technologies this activity may be divided into two parts: the first external to the firm, representing the acquisition of investment goods, denoted  $A(t)$ ; and the second internal to the firm, representing the installation of investment goods. The internal activities of the firm include the production of output and the installation of investment goods. We assume that capital is allocated between production and installation; the higher the rate of acquisition of investment goods, the larger the amount of capital that must be withheld from production. Flows of investment goods acquisition and installation are measured as the number of machines acquired or installed.

To formalize our description of recursive technologies, we first suppose that the set of production possibilities may be represented in the form,

$$(1) \quad \begin{aligned} Q(t) &\leq F[K_F(t), L(t)], \\ Q(t) &\geq 0, K_F(t) \geq 0, L(t) \geq 0, \end{aligned}$$

where  $F$  is the maximum level of output corresponding to the amount of capital input allocated to production  $K_F(t)$  and the amount of labor input  $L(t)$ . We assume that output, capital input, and labor input are nonnegative and that the production process is

2. A. P. Lerner, "Capital, Investment, and Interest," in *The Economics of Control* (New York: The Macmillan Company, 1944), Ch. 25, pp. 323-45.

3. R. Eisner and R. H. Strotz, "Determinants of Business Investment," in Commission on Money and Credit, *Impacts of Monetary Policy* (Englewood Cliffs, N.J.: 1962), Section I, "The Theoretical Framework," pp. 61-116.

4. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, trans. K. N. Trirgoff (New York: Interscience Publishers, 1962).

characterized by free disposal. Any nonnegative level of production  $Q(t)$  less than or equal to the maximum given by the production function  $F$  is feasible.

Similarly, we may represent the set of installation possibilities in the form,

$$(2) \quad \begin{aligned} A(t) &\leq I[K_F(t), K_I(t)], \\ A(t) &\geq 0, K_I(t) \geq 0, \end{aligned}$$

where  $I$  is the maximum level of acquisition of investment goods corresponding to the amounts of capital input allocated to production  $K_F(t)$  and to installation  $K_I(t)$ . We assume that acquisition of investment goods and capital input allocated to installation are nonnegative and that the installation process is characterized by free disposal. Any nonnegative level of acquisition  $A(t)$  less than or equal to the maximum given by the installation function  $I$  is feasible. Finally, the total capital input available  $K(t)$  is greater than or equal to the sum of capital input allocated to production and to installation:

$$(3) \quad K(t) \geq K_F(t) + K_I(t).$$

The allocation of capital is also characterized by free disposal.

We assume that the production function  $F$  is twice differentiable and increasing, so that marginal products of labor and capital services are positive:

$$(4) \quad \frac{\partial F}{\partial K_F} > 0, \frac{\partial F}{\partial L} > 0.$$

Further, we assume that the production function is concave; production is characterized by nonincreasing returns and a decreasing marginal rate of substitution. The second-order derivatives satisfy the conditions,

$$(5) \quad \frac{\partial^2 F}{\partial K_F^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0; \frac{\partial^2 F}{\partial K_F^2} \frac{\partial^2 F}{\partial L^2} \geq \left( \frac{\partial^2 F}{\partial K_F \partial L} \right)^2.$$

If the last inequality is satisfied with strict equality, the production function is characterized by constant returns.

We assume that the installation function  $I$  is twice differentiable and increasing, so that marginal products of capital allocated to production and installation are positive:

$$(6) \quad \frac{\partial I}{\partial K_F} > 0, \frac{\partial I}{\partial K_I} > 0.$$

Further, we assume that the installation function is concave; installation is characterized by nonincreasing returns and a decreasing

marginal rate of substitution. The second-order derivatives satisfy the conditions,

$$(7) \quad \frac{\partial^2 I}{\partial K_F^2} < 0, \quad \frac{\partial^2 I}{\partial K_I^2} < 0; \quad \frac{\partial^2 I}{\partial K_F^2} \frac{\partial^2 I}{\partial K_I^2} \geq \left( \frac{\partial^2 I}{\partial K_F \partial K_I} \right)^2.$$

If the last inequality is satisfied with strict equality, the installation function is characterized by constant returns.

Our characterization of technology can be extended in a number of directions. First, the production and installation processes could be combined into an overall substitution law:

$$G[Q(t), A(t), K(t), L(t)] = 0.$$

Output and the acquisition of investment goods are joint products, and labor and capital services are factors of production. Under our assumption that production and installation are separable activities, the analysis is considerably simplified. Separable processes have been analyzed by Lucas, Uzawa, Gould, and Treadway.<sup>5</sup> Treadway has studied nonseparable processes.<sup>6</sup> Second, the installation process could depend on the rate of change of capital rather than the rate of investment.<sup>7</sup> Third, production and installation could be characterized by nonnegative and nondecreasing marginal products. The marginal rate of substitution of labor for capital input in production could be made nonincreasing, and the marginal rate of substitution of capital for installation could be made nonincreasing. All of these extensions would introduce inessential detail into our presentation. Only dropping the assumption of separability of the processes of production and installation produces qualitatively different results.

To simplify our presentation further, it is useful to add conditions that assure the existence of an interior solution to the usual first-order conditions for a maximum of profit in the absence of the internal installation process. For all positive  $K_F$  we assume that the marginal product of labor input has limits:

$$(8) \quad \lim_{L \rightarrow 0} \frac{\partial F}{\partial L} = +\infty, \quad \lim_{L \rightarrow +\infty} \frac{\partial F}{\partial L} = 0;$$

similarly, we assume that for all positive  $L$ ,

5. Lucas, *op. cit.*; Uzawa, *op. cit.*; J. P. Gould, "Adjustment Costs in the Theory of Investment of the Firm," *Review of Economic Studies*, XXXV (Jan. 1968), 47-55; and A. B. Treadway, "On Rational Entrepreneurial Behaviour and the Demand for Investment," *Review of Economic Studies*, XXXVI (April 1969), 227-39.

6. A. B. Treadway, "On a Dynamic Theory of the Competitive Firm and Its Implications for Propositions in Static Theory," unpublished manuscript (April 1969), 62 pp.

7. Treadway, *op. cit.*, employs the rate of change of capital; Lucas, *op. cit.*, and Gould, *op. cit.*, employ the rate of investment.



$$(9) \quad \lim_{K_F \rightarrow 0} \frac{\partial F}{\partial K_F} = +\infty, \quad \lim_{K_F \rightarrow +\infty} \frac{\partial F}{\partial K_F} = 0.$$

We impose analogous conditions on the installation process, so that for all positive  $K_I$  we assume that the marginal product of capital input allocated to production has limits:

$$(10) \quad \lim_{K_F \rightarrow 0} \frac{\partial I}{\partial K_F} = +\infty, \quad \lim_{K_F \rightarrow +\infty} \frac{\partial I}{\partial K_F} = 0;$$

similarly, we assume that for all positive  $K_F$ ,

$$(11) \quad \lim_{K_I \rightarrow 0} \frac{\partial I}{\partial K_I} = +\infty, \quad \lim_{K_I \rightarrow +\infty} \frac{\partial I}{\partial K_I} = 0.$$

For *decomposable* technologies the acquisition of investment goods is unconstrained. No capital is required for the installation of investment goods. The level of acquisition may be positive, negative, or zero. We can specialize the description of technology outlined above by eliminating the installation function and non-negativity of the acquisition of investment goods (2) as constraints and by setting the value of capital allocated to installation  $K_I$  equal to zero. We retain the production function (1) as characterized by assumptions (4) and (5) as a constraint. Under this specialization our description of technology at each point of time reduces to the *classical* technology of the atemporal theory of production. Accordingly, we may refer to our description of technology with constraints on the acquisition of investment goods as *nonclassical*.

### 3. OBJECTIVE FUNCTION

For intertemporal choice the objective of economic activity can be described as the selection of a most preferred consumption plan, subject to technology and to economic possibilities for transformation of the results of production into consumption. This optimization problem involves the selection of a production plan and a consumption plan. Given fixed prices, the optimization problem is recursive in structure. The productive unit, firm or industry, selects a production plan that maximizes its value. The resulting maximum value is then taken as given for the determination of an optimal consumption plan. The choice of a production plan is completely independent of the choice of a consumption plan. We may identify the selection of a production plan with the theory of producer behavior and the selection of a consumption plan with the theory of consumer behavior.

The intertemporal interpretation of the recursive structure of the problem of optimal consumption and production planning under fixed prices is due to Fisher.<sup>8</sup> Recursive decomposition of production and consumption planning is not specific to the intertemporal interpretation of atemporal economic theory, but is characteristic of economic decisions at the most abstract theoretical level, as Koopmans has emphasized.<sup>9</sup> The full implications of this recursive decomposition for intertemporal production planning have been appreciated only recently, largely as a consequence of the revival and extension of Fisher's analysis of the investment decision by Hirshleifer and Bailey.<sup>1</sup>

From one point of view the objective of maximizing the value of the firm is only one of many possible criteria for production and investment decisions.<sup>2</sup> Alternative criteria discussed in the literature include maximization of the average internal rate of return, maximization of the rate of return on capital owned by the firm, investment in any project with internal rate of return greater than the market rate of interest, and so on. None of these criteria can be derived as a necessary condition for selection of an optimal consumption plan under the conditions we have outlined. Hirshleifer has summarized the justification for Fisher's approach as follows:

Since Fisher, economists working in the theory of investment decision have tended to adopt a mechanical approach—some plumping for the use of this formula, some for that. From a Fisherian point of view, we can see that none of the formulas so far propounded is universally valid. Furthermore, even where the present-value rule, for example, is correct, few realize that its validity is conditional upon making certain associated financing decisions as the Fisherian analysis demonstrates. In short, the Fisherian approach permits us to define the range of applicability and the short-comings of all the proposed formulas—thus standing over against them as the general theoretical solution to the problem of investment decision under conditions of certainty.<sup>3</sup>

8. Fisher, *op. cit.*, p. 148.

9. T. C. Koopmans, "Allocation of Resources and the Price System," in T. C. Koopmans, *Three Essays on the State of Economic Science* (New York: McGraw-Hill, 1957), pp. 1-126, esp. pp. 16-21.

1. J. Hirshleifer, "On the Theory of the Optimal Investment Decision," in E. Solomon, ed., *The Management of Corporate Capital* (Glencoe, Ill.: The Free Press of Glencoe, 1959), pp. 205-28; M. J. Bailey, "Formal Criteria for Investment Decisions," *Journal of Political Economy*, LXVII (Oct. 1959), 476-88; and J. Hirshleifer, *Investment, Interest, and Capital* (Englewood Cliffs, N.J.: Prentice-Hall, 1970).

2. In a survey paper on the theory of capital Lutz remarks: "It is one of the surprising things about capital theory that no agreement seems to have been reached as to what the entrepreneur should maximize." F. A. Lutz, "The Essentials of Capital Theory," in F. A. Lutz and D. Hague, eds., *The Theory of Capital* (London: Macmillan, 1961), p. 6.

3. J. Hirshleifer, *op. cit.*, p. 228.

A basic difficulty with the usual intertemporal interpretation of the theory of production, as in Fisher,<sup>4</sup> is that transactions in present and all future commodities are viewed as taking place in the present, ostensibly requiring futures markets for all commodities in all time periods. But are the specified future deliveries actually made or are contracts modified as a given future time point approaches the present? If the contracts are modified, why do individuals agree to contracts at the terms originally stipulated? Hicks has altered Fisher's interpretation by identifying future transactions with hypothetical plans that do not result in actual contracts. These plans are constructed in accord with expectations of future prices. Plans may be modified as expectations change.<sup>5</sup>

In the intertemporal interpretation of economic theory developed by Fisher and Hicks, a "horizon" is imposed on producers so that only a finite interval of time is considered. The imposition of a finite time horizon is inappropriate for analyzing the movement of an economic system through time. The selection of a production plan in the present involves decisions about the future based on maximization of the value of the producing unit in the present. To be consistent with future decisions, the present production plan must be based on the same technological and economic possibilities as future plans. If a finite horizon is imposed on present decisions, the horizon must remain fixed in time so that as time proceeds, the horizon becomes shorter and shorter, introducing a complete asymmetry between present and future decisions. To overcome this difficulty, it is useful to introduce an unlimited time horizon, as in Malinvaud's intertemporal interpretation of economic theory.<sup>6</sup>

In the absence of a time horizon, the value of the producing unit, industry or firm, at any point of time is the integral of the value of revenue less outlay on current account, less outlay on capital account over current and future points of time. Outlay on capital account includes costs of acquisition of investment goods. We will refer to the value of revenue less outlay on current account as the value of *cash flow* and the value of revenue less outlay on both current and capital account as *net cash flow*, defined by

$$(12) \quad p(t)C(t) = P_Q(t)Q(t) - p_L(t)L(t) - p_A(t)A(t),$$

where  $p(t)$  is the present price of money or discount factor and  $r(t)$  is the rate of interest:

4. Fisher, *op. cit.*, pp. 302-10.

5. Hicks, *op. cit.*, pp. 124-26, 202-26.

6. Malinvaud, *op. cit.*, pp. 242-51.

$$p(t) = e^{-\int_0^t r(s) ds},$$

and  $p_Q(t)$ ,  $p_L(t)$ , and  $p_A(t)$  are present prices of output, labor services, acquisition of investment goods, respectively. To simplify the presentation, we assume that all present prices are positive. Interpreting cash flow in terms of future prices, we may write

$$(13) \quad C(t) = q_Q(t)Q(t) - q_L(t)L(t) - q_A(t)A(t),$$

where  $p_Q(t) = p(t)q_Q(t)$ , and so on.

The value of the producing unit at time  $s$  is the integral of the value of net cash flow from  $s$  forward, defined by

$$(14) \quad p(s)V(s) = \int_s^\infty p(t)C(t)dt.$$

The *present value* of the producing unit is its value in the present:

$$(15) \quad p(O)V(O) = \int_0^\infty p(t)C(t)dt,$$

of course,  $p(O) = 1$ , so that we may refer to  $V(O)$  as present value. An obvious property of this objective function is that present value of a production plan is equal to the integral of the value of net cash flow from the present to some future time, say  $t$ , plus the present value of the producing unit at  $t$ .

Under recursive decomposition of production and consumption planning, the objective of the producing unit is to maximize its present value, subject to the past time path for capital accumulation. If the producing unit follows an optimal production and investment plan from the present to some future time, say  $t$ , the optimal plan from  $t$  forward maximizes the current present value at  $t$ , subject to the time path for accumulation of capital up to that point. Present and future decisions are *consistent* in the sense of Strotz.<sup>7</sup> The optimal production plan at each point of time may be generated by maximizing the current present value of the producing unit, subject to the past time path for capital accumulation.

Before deriving the optimal plan for production and investment, it is useful to analyze further the acquisition and replacement of investment goods. We may write the value of the producing unit at time  $s$  in the form,

7. R. H. Strotz, "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, XXIII (Jan. 1956), 165-80.

$$\begin{aligned}
 p(s)V(s) &= \int_s^\infty [p_Q(t)Q(t) - p_L(t)L(t) - p_A(t)A(t)] dt, \\
 &= \int_s^\infty [p_Q(t)Q(t) - p_L(t)L(t)] dt \\
 &\quad - \int_s^\infty p_A(t)A(t) dt,
 \end{aligned}$$

provided, of course, that all present values are defined.

We may define the value of revenue less outlay on current account and the time rental value of capital services as the value of *profits*, defined by

$$(16) \quad p(t)P(t) = p_Q(t)Q(t) - p_L(t)L(t) - p_K(t)K(t).$$

The *goodwill* of the producing unit at time  $s$  is the integral of the value of profits from  $s$  forward, defined by

$$(17) \quad p(s)V^+(s) = \int_s^\infty p(t)P(t) dt.$$

The present value of goodwill of the producing unit is its value in the present:

$$(18) \quad p(O)V^+(O) = \int_0^\infty p(t)P(t) dt;$$

since  $p(O)=1$ , we may refer to  $V^+(O)$  as the *present value of goodwill*.

Returning to our expression for the value of the producing unit, we may write

$$\begin{aligned}
 (19) \quad p(s)V(s) &= \int_s^\infty [p_Q(t)Q(t) - p_L(t)L(t) - p_K(t)K(t)] dt \\
 &\quad + \int_{-\infty}^s p_A(s,t)A(t) dt, \\
 &= p(s)V^+(s) + \int_{-\infty}^s p_A(s,t)A(t) dt.
 \end{aligned}$$

The value of the producing unit is equal to the value of goodwill plus the value of the firm's capital stock, where  $p_A(s,t)$  is the present price of a capital good first acquired at time  $t$ . This formula for the value of the producing unit is due to Malinvaud.<sup>8</sup>

For exponential decline in efficiency we may write the value of capital stock in the form,

$$\begin{aligned}
 \int_s^\infty p_K(u) \int_{-\infty}^s A(t)e^{-\delta(u-t)} dt du, \\
 = \int_s^\infty p_K(u)e^{-\delta(u-s)} \int_{-\infty}^s A(t)e^{-\delta(s-t)} dt du, \\
 = p_A(s)K(s).
 \end{aligned}$$

The value of capital stock is equal to the acquisition price of capital

8. Malinvaud, *op. cit.*, p. 253.

goods multiplied by the quantity of capital. The value of the producing unit may be expressed in the form,

$$(20) \quad p(s)V(s) = p(s)V^+(s) + p_A(s)K(s),$$

or in future prices,

$$(21) \quad V(s) = V^+(s) + q_A(s)K(s).$$

#### 4. OPTIMAL INVESTMENT POLICY

The objective of the producing unit, firm or industry, is to maximize present value (15), defined as

$$V(O) = \int_0^\infty [p_Q(t)Q(t) - p_L(t)L(t) - p_A(t)A(t)] dt,$$

subject to the production function (1),

$$Q(t) \leq F[K_F(t), L(t)],$$

the installation function (2),

$$A(t) \leq I[K_F(t), K_I(t)],$$

and the identity for the allocation of capital (3),

$$K(t) \geq K_F(t) + K_I(t),$$

for nonnegative levels of output  $Q(t)$ , labor input  $L(t)$ , and acquisition of investment goods  $A(t)$ , and nonnegative levels of capital for production  $K_F(t)$  and for installation  $K_I(t)$ . To simplify the discussion, we assume that the decline in relative efficiency of investment goods with age is exponential,

$$K'(t) = A(t) - \delta K(t),$$

so that current replacement requirements depend on the current level of capital stock and are independent of past levels. Given the initial level of capital stock  $K(O)$ , this is a problem in optimal control theory. To characterize the optimal policy, we employ Pontryagin's maximum principle.<sup>9</sup>

Interpreting the determination of investment policy as a problem in control theory, we may identify output  $Q(t)$ , labor input  $L(t)$ , acquisition of investment goods  $A(t)$ , and capital allocated to production  $K_F(t)$  and installation  $K_I(t)$  as control variables or instruments. Similarly, we may identify the level of capital stock  $K(t)$  as a state variable or target.<sup>1</sup> At this point we may rewrite the present value of the producing unit in terms of future prices:

9. See note 4, p. 526.

1. The terms "instruments" and "targets" are employed by J. Tinbergen, *Economic Policy: Principles and Design* (Amsterdam: North-Holland, 1956), pp. 3-10.

$$V(O) = \int_0^{\infty} p(t) [q_Q(t)Q(t) - q_L(t)L(t) - q_A(t)A(t)] dt,$$

where the present value of money  $p(t)$  serves as a discount factor.

To employ Pontryagin's maximum principle for the characterization of an optimal investment policy we introduce the Hamiltonian function,

$$(22) \quad p(t) \mathfrak{H}(K; Q, L, A, K_F, K_I; \lambda_A; t) \\ = p(t) (q_Q Q - q_L L - q_A A) + p(t) \lambda_A(t) (A - \delta K),$$

where  $\lambda_A(t)$  is a multiplier associated with the acquisition and installation of investment goods at time  $t$ . This multiplier represents the discounted value of all future increments in cash flow associated with an increment in investment at time  $t$ . We discount this multiplier to the present in order to facilitate its economic interpretation; future prices are discounted to the present in the same way. The multiplier  $\lambda_A(t)$  represents the value of acquisition and installation of a unit of investment goods at time  $t$ ; accordingly, we refer to this multiplier as the *shadow asset price of capital*. Given the shadow asset price of capital for the optimal investment policy, the Hamiltonian must be a maximum, subject to the production function, the installation function, and the identity for the allocation of capital.<sup>2</sup> To characterize maxima of the Hamiltonian, we employ the Kuhn-Tucker theorem of concave programming.<sup>3</sup>

By the Kuhn-Tucker theorem, maxima of the Hamiltonian subject to constraint for a given value of the multiplier  $\lambda_A(t)$  correspond to saddlepoints of the Lagrangean function,

$$(23) \quad p(t) \mathfrak{L}(K; Q, L, A, K_F, K_I; \lambda_A; \mu_Q, \mu_I, \mu_K; t) \\ = p(t) \mathfrak{H} + p(t) \mu_Q(t) (Q(K_F, L) - Q) \\ + p(t) \mu_I(t) (I(K_F, K_I) - A) \\ + p(t) \mu_K(t) (K - K_F - K_I).$$

The multiplier  $p(t) \mu_Q(t)$  represents the increment in the value of the Hamiltonian associated with an increment in output at time  $t$ , holding levels of capital and labor input constant. Similarly, the multiplier  $p(t) \mu_I(t)$  represents the increment in the value of the Hamiltonian associated with an increment in the installation of capital at time  $t$ , holding levels of capital for production and capital for installation constant. Finally, the multiplier  $p(t) \mu_K(t)$  repre-

2. Our application of Pontryagin's maximum principle follows the exposition of K. J. Arrow and M. Kurz, *Public Investment, the Rate of Return, and Optimal Fiscal Policy* (Baltimore: Johns Hopkins Press, 1970), pp. 33-51.

3. H. W. Kuhn and A. W. Tucker, "Non-Linear Programming," in J. Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley: University of California Press, 1950), pp. 481-92.

sents the increment in the value of the Hamiltonian associated with an increment in capital stock at time  $t$ . We refer to  $\mu_Q$  as the *shadow price of output*,  $\mu_I$  as the *shadow price of installation of investment goods*, and  $\mu_K$  as the *shadow rental price of capital*. The multipliers  $\{p\mu_Q, p\mu_I, p\mu_K\}$  are equal to the shadow prices  $\{\mu_Q, \mu_I, \mu_K\}$ , discounted to the present; this convention facilitates comparisons between the shadow prices and future market prices.

A necessary condition for an optimal investment policy is that at each point of time the shadow asset price of capital satisfies

$$(24) \quad \lambda'_A = \lambda_A r - \frac{\partial \mathfrak{L}}{\partial K}, \\ = \lambda_A r + \lambda_A \delta - \mu_K.$$

The shadow rental price of capital is equal to the shadow asset price of capital, multiplied by the sum of the rate of interest and the rate of depreciation, less the rate of change of the shadow asset price of capital,

$$\mu_K = \lambda_A (r + \delta) - \lambda'_A.$$

To characterize saddlepoints of the Lagrangean function  $\mathfrak{L}$  at each point of time, we first observe that the Hamiltonian function for investment policy is linear in the instruments  $\{Q, L, A, K_F, K_I\}$ . The identity for the allocation of capital is linear, and the production and installation functions are concave in these instruments. We conclude that for given values of capital  $K$  and the shadow asset price of capital  $\lambda_A$ , the objective function and the constraints are concave. Provided that for any set of shadow prices  $\{\mu_Q, \mu_I, \mu_K\}$ , nonnegative and not all zero, the constraint qualification,

$$\mu_Q (F(K_F, L) - Q) + \mu_I (I(K_F, K_I) - A) \\ + \mu_K (K - K_F - K_I) > 0,$$

is satisfied for some set of instruments  $\{Q, L, A, K_F, K_I\}$ , any maximum of the Hamiltonian corresponds to a saddlepoint of the Lagrangean.<sup>4</sup> If output, capital, and investment goods are freely disposable, so that for some feasible investment policy,

$$Q < F(K_F, L), \\ A < I(K_F, K_I), \\ K_I + K_F < K,$$

then the constraint qualification is satisfied.

For given values of capital  $K$  and the shadow asset price of

4. This form of the constraint qualification is employed by S. Karlin, *Mathematical Methods and Theory in Games, Programming and Economics*, I (Reading, Mass.: Addison-Wesley Publishing Company, 1959), p. 201.



capital  $\lambda_A$ , a saddlepoint of the Lagrangean function implies the differential inequalities,

$$(25) \quad \begin{aligned} \frac{\partial \mathcal{L}}{\partial \mu_Q} &= F(K_F, L) - Q \geq 0, \\ \frac{\partial \mathcal{L}}{\partial \mu_I} &= I(K_F, K_I) - A \geq 0, \\ \frac{\partial \mathcal{L}}{\partial \mu_K} &= K - K_F - K_I \geq 0, \end{aligned}$$

and the complementary slackness condition,

$$(26) \quad \mu_Q \frac{\partial \mathcal{L}}{\partial \mu_Q} + \mu_I \frac{\partial \mathcal{L}}{\partial \mu_I} + \mu_K \frac{\partial \mathcal{L}}{\partial \mu_K} = 0,$$

for nonnegative multipliers  $\{\mu_Q, \mu_I, \mu_K\}$ . Further, a saddlepoint of the Lagrangean implies the differential inequalities,

$$(27) \quad \begin{aligned} \frac{\partial \mathcal{L}}{\partial Q} &= q_Q - \mu_Q \leq 0, \\ \frac{\partial \mathcal{L}}{\partial L} &= -q_L + \mu_Q \frac{\partial F}{\partial L} \leq 0, \\ \frac{\partial \mathcal{L}}{\partial A} &= -q_A + \lambda_A - \mu_I \leq 0, \\ \frac{\partial \mathcal{L}}{\partial K_F} &= \mu_Q \frac{\partial F}{\partial K_F} + \mu_I \frac{\partial I}{\partial K_F} - \mu_K \leq 0, \\ \frac{\partial \mathcal{L}}{\partial K_I} &= \mu_I \frac{\partial I}{\partial K_I} - \mu_K \leq 0; \end{aligned}$$

and the complementary slackness condition,

$$(28) \quad Q \frac{\partial \mathcal{L}}{\partial Q} + L \frac{\partial \mathcal{L}}{\partial L} + A \frac{\partial \mathcal{L}}{\partial A} + K_F \frac{\partial \mathcal{L}}{\partial K_F} + K_I \frac{\partial \mathcal{L}}{\partial K_I} = 0,$$

for nonnegative instruments  $\{Q, L, A, K_F, K_I\}$ .<sup>5</sup>

## 5. DECISION RULES

To characterize the decision rules associated with an optimal investment policy, we now proceed to a detailed analysis of the saddlepoint conditions — (25), (26), (27), (28) — for the Lagrangean function (23). At each point of time we take the value of capital  $K$  and its shadow price  $\lambda_A$  as given by the optimal invest-

5. *Ibid.*, pp. 203–04.

ment policy up to that point. From the differential inequality for output (27) we first observe that

$$\mu_Q \geq q_Q > 0,$$

by our assumption that prices are positive. If the price of output is positive, the shadow price of output is positive, and free disposal of output does not pay. By complementary slackness (26) we may replace the inequality constraint (25) on output by the strict equality.

$$(29) \quad Q = F(K_F, L).$$

Second, for  $\mu_Q > 0$  the differential inequality (27),

$$\mu_Q \frac{\partial F}{\partial L} \leq q_L,$$

can be satisfied only if labor input is positive,  $L > 0$ . But this implies that output is positive,  $Q > 0$ . By complementary slackness (28), the inequalities we have considered up to this point become equalities:

$$(30) \quad q_Q = \mu_Q,$$

$$q_Q \frac{\partial F}{\partial L} = q_L,$$

the shadow price of output is equal to its market price and the value of the marginal product of labor is equal to the wage rate.

Turning to the differential inequality (27) for capital allocated to production  $K_F$ , we observe that a positive price of output implies that the constraint,

$$q_Q \frac{\partial F}{\partial K_F} + \mu_I \frac{\partial I}{\partial K_F} \leq \mu_K,$$

can be satisfied only if capital allocated to production is positive,  $K_F > 0$ . By complementary slackness (28) the inequality becomes an equality,

$$(31) \quad q_Q \frac{\partial F}{\partial K_F} + \mu_I \frac{\partial I}{\partial K_F} = \mu_K.$$

The value of the marginal product in both production and installation of capital allocated to production is equal to the shadow rental price of capital. The value in production is a market value, measured at the market price of output  $q_Q$ . The value in installation is an imputed value, measured at the shadow price of installation of investment goods  $\mu_I$ . Since the shadow rental price of capital is positive,  $\mu_K > 0$ , complementary slackness (26) implies that installed capital is fully utilized,

$$(32) \quad K = K_F + K_I.$$

Free disposal of capital does not pay.

At this point we must consider two cases. First, the shadow price of installation of investment goods  $\mu_I$  may be zero. In this case no capital is allocated to installation, since

$$(33) \quad 0 = \mu_I \frac{\partial I}{\partial K_I} < \mu_K,$$

the value of the marginal product of capital allocated to installation is less than the shadow rental price of capital. All capital is allocated to production, and the acquisition of capital is zero. The shadow asset price of capital is less than or equal to its market acquisition price (27),

$$\lambda_A \leq q_A.$$

Second, the price of installation of new investment goods may be positive,  $\mu_I > 0$ . In this case the differential inequality for capital allocated to installation (27) can be satisfied only if capital allocated to installation is positive,  $K_I > 0$ . By complementary slackness (28) this inequality becomes an equality,

$$(34) \quad \mu_I \frac{\partial I}{\partial K_I} = \mu_K;$$

the value of the marginal product of capital allocated to installation is equal to the shadow rental price of capital. By complementary slackness (26), the constraint on the installation of capital (25) is satisfied with strict equality,

$$(35) \quad A = I(K_F, K_I).$$

Free disposal of investment goods does not pay. Since levels of capital allocated to production and installation are both positive, acquisition of investment goods is positive. By complementary slackness (28),

$$(36) \quad \lambda_A = q_A + \mu_I,$$

the shadow asset price of capital is equal to the market price of acquisition plus the shadow price of installation.

We have presented necessary conditions for an optimal investment policy. To complete the characterization of an optimal policy, we employ a set of sufficient conditions consisting of the necessary conditions given above and the transversality conditions,

$$(37) \quad \lim_{t \rightarrow \infty} p(t) \lambda_A(t) \geq 0, \quad \lim_{t \rightarrow \infty} p(t) \lambda_A(t) K(t) = 0.$$

The first of these conditions states that the limiting value of the discounted shadow asset price of capital must be nonnegative. The second states that the limiting value of capital, evaluated at its dis-

counted shadow asset price, must be equal to zero.<sup>6</sup> This condition requires, for example, that if the shadow price  $\lambda_A(t)$  is constant, capital  $K(t)$  is growing at a constant rate, and the rate of interest  $r(t)$ , equal to the rate of decline of the discount factor  $p(t)$ , is constant, then the rate of interest must exceed the rate of growth of capital. If there exists an investment policy that satisfies the differential equations for capital and its shadow asset price, the saddle-point conditions for the Lagrangean at each point of time and the transversality conditions, then this policy is optimal.<sup>7</sup>

At any point of time, the target  $K$  and the multiplier  $\lambda_A$  are determined by the optimal investment policy up to that point; the values of these variables depend on the initial condition for capital and the terminal condition for the shadow asset price derived from the transversality conditions. For given values of capital and its shadow asset price, a set of values for the instruments  $\{Q, L, A, K_F, K_I\}$  and the Lagrange multipliers  $\{\mu_Q, \mu_I, \mu_K\}$  are determined from the saddlepoint conditions for the Lagrangean function  $p\mathcal{L}$ . Given optimal values of the instruments and the Lagrange multipliers, optimal rates of change of capital and its shadow asset price are determined from the differential equations,

$$\begin{aligned} K' &= A - \delta K, \\ \lambda_A' &= \lambda_A(r + \delta) - \mu_K. \end{aligned}$$

The optimal production and investment plan at any point of time can be generated by maximizing the current present value of the productive unit, subject to technology and to the level of capital accumulated up to that point. The optimal plan depends on the level of capital and the value of the optimal shadow asset price of capital. This plan determines the level of investment and the rate of change of capital, together with the rate of change of the shadow asset price. The resulting decision rules are *recursive*, since they generate a new time path for capital and its shadow asset price and new decision rules of the same form.

The optimal investment plan for recursive technologies depends on the level of capital services required for the optimal production plan at each point of time and on replacement requirements. Replacement requirements depend only on the level of capital stock and are otherwise independent of the time path for capital accumulation if and only if the efficiency of investment goods declines exponentially with age. Under this condition decision rules for the optimal investment plan are *Markovian* in the sense that the cur-

6. Arrow and Kurz, *op. cit.*, pp. 43-51.

7. *Ibid.*, p. 49.

rent stock of capital contains all the information about past decisions that is relevant to the future. The optimal production and investment plan at any point of time depends on accumulated capital stock, but is otherwise independent of past production and investment decisions.

The acquisition of capital goods is unconstrained if no capital is required for the installation of investment goods, and the level of acquisition may be positive, negative, or zero. We first specialize the model outlined above by eliminating the installation function (2) as a constraint and setting the corresponding Lagrange multiplier  $\mu_I$ , the shadow price of installation of investment goods, equal to zero. We retain the nonnegativity constraint on the level of acquisition; the resulting technology is recursive. If the level of acquisition is positive, the shadow asset price of capital is equal to its market acquisition price (27),

$$\lambda_A = q_A.$$

The marginal productivity conditions (27) reduce to

$$q_Q \frac{\partial F}{\partial L} = q_L,$$

$$q_Q \frac{\partial F}{\partial K} = q_K,$$

where

$$q_K = q_A(r + \delta) - q'_A$$

is the market rental price of capital. The value of the marginal product of capital is equal to its market rental price, and the value of the marginal product of labor is equal to its market wage rate.

If the acquisition of investment goods is zero, the shadow asset price of capital is less than or equal to the market acquisition price (27),

$$\lambda_A \leq q_A.$$

The marginal productivity condition for capital (27) becomes

$$q_Q \frac{\partial F}{\partial K} = \mu_K,$$

where

$$\mu_K = \lambda_A(r + \delta) - \lambda'_A$$

is the shadow rental price of capital (26). The value of the marginal product of capital is equal to its shadow rental price; as before, the value of the marginal product of labor is equal to its market wage rate.

First, we conclude that at every point of time the necessary

conditions for an optimal production plan of the atemporal theory of production are satisfied or the level of acquisition of investment goods is zero. If this level is zero, the marginal productivity condition for capital of atemporal theory is replaced by an analogous condition with the shadow rental price  $\mu_K$  in place of the market rental price  $q_K$ . The resulting decision rules are recursive in the sense that the optimal plan depends on the level of capital and the value of the optimal shadow asset price of capital. This plan generates a new time path for capital and its shadow asset price and new decision rules of the same form. This model of investment policy has been analyzed in greater detail by Arrow.<sup>8</sup>

Second, we specialize the model outlined above by eliminating both the installation function and the nonnegativity constraint on the level of acquisition of investment goods (2). The resulting technology is decomposable in the sense that production possibilities at any point of time are independent of past productive activity, including accumulated stocks. The inequality constraint (27) relating the shadow asset price of capital to its market acquisition price becomes an equality,

$$\lambda_A = q_A,$$

so that the marginal productivity conditions reduce to the necessary conditions for an optimal production plan of the atemporal theory of production, as given above. The value of the marginal product of capital is equal to its market rental and the value of the marginal product of labor is equal to its market wage.

As before, the optimal production and investment plan at any point of time can be generated by maximizing the current present value of the productive unit at that point, subject to technology and the level of capital accumulated up to that point. Given the initial level of capital, this is equivalent to maximizing the current present value of goodwill at each point of time, subject to technology. For decomposable technologies there is no restriction on the acquisition of investment goods. The optimal production plan is independent of the past time path of capital accumulation. Production decisions at each point of time are independent of decisions at other points of time; investment decisions are related only through replacement requirements. The optimal production plan at any point of time can be generated by maximizing the current level of profit.

Profit depends on the price of output, the wage rate, and the

8. See note 9, p. 525.

rental price of capital services. The rental price of capital at each point of time depends on the current price and own rate of interest of investment goods if and only if the decline in relative efficiency of investment goods with age is exponential. In this case the decision rules that result from profit maximization are independent both of current capital stock and of all future prices and rates of interest. Current profit depends only on current prices of output, labor services, and investment goods, and the current own rate of interest on investment goods. Under these conditions decision rules associated with profit maximization are *myopic* in the sense of Strotz.<sup>9</sup> Myopic decision rules for durable goods with exponential decline in relative efficiency are derived from present value maximization by Haavelmo.<sup>1</sup>

Decision rules that result from maximizing current present value at each point of time are *decomposable* in the sense that they are independent of decision rules at other points of time. These are the usual decision rules of the atemporal theory of production. This model of investment policy has been analyzed in greater detail by Jorgenson.<sup>2</sup> Decomposable decision rules associated with profit maximization at each point of time are derived from present value maximization by Malinvaud.<sup>3</sup> In earlier treatments of the durable goods model, for example by Walras,<sup>4</sup> the decomposable decision rule of marginal productivity theory had been employed without deriving it as an implication of present value maximization.

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9. Strotz, *op. cit.*

1. T. Haavelmo, *A Study in the Theory of Investment* (Chicago: University of Chicago Press, 1960), Ch. 28, "The Individual Producer's Demand for Capital," pp. 161-70, esp. p. 163.

2. D. W. Jorgenson, "The Theory of Investment Behavior," in R. Ferber, ed., *The Determinants of Investment Behavior* (New York: Columbia University Press, 1967), pp. 129-56.

3. Malinvaud, *op. cit.*, p. 253.

4. Walras, *op. cit.*, pp. 382-86.